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Edge Hubtic Number in Graphs

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Abstract: The maximum order of partition of the edge set $E(G)$ into edge hub sets is called edge hubtic number of G and denoted by $\xi_e(G)$. In this paper, we determine the edge hubtic number of some standard graphs. Also we obtain bounds for $\xi_e(G)$. In addition we characterize the class of all (p, q) graphs for which $\xi_e(G) = q$.

Key Words: Edge hubtic number, edge hub number, partition.

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§1. Introduction

By a graph $G = (V, E)$, we mean a finite and undirected graph without loops and multiple edges. A graph G with p vertices and q edges is called a (p, q) graph, the number p is referred to as the order of a graph G and q is referred to as the size of a graph G. In general, the degree of a vertex v in a graph G denoted by $deg(v)$ is the number of edges of G incident with v. The degree of an edge uv is defined to be $deg(u) + deg(v) - 2$. Also $\Delta^{'}(G)$ denotes the maximum degree among the edges of G, and $\delta^{'}(G)$ denotes the minimum degree among the edges of G. $\lfloor x \rfloor$ is the greatest integer less than or equal to x. In a tree, a leaf is a vertex of degree one, a leaf edge is an edge incident to a leaf. We refer to [6] for terminology and notations not defined here.

Introduced by Walsh [13], a hub set in a graph G is a set H of vertices in G such that any two vertices outside H are connected by a path whose internal vertices lie in H. The hub number of G, denoted by $h(G)$, is the minimum size of a hub set in G. A connected hub set in G is a vertex hub set F such that the subgraph of G induced by F (*denoted* $G[F]$) is connected.

Let G be a graph, let $e = (u, v)$ and $f = (u_1, v_1)$, a path between two edges e and f is a path between one end vertex from e and another end vertex from f such that $d(e, f) = min\{d(u, u_1), (u, v_1), (v, u_1), (v, v_1)\}$. Internal edges of a path between two edges e and f are all the edges of the path except e and f [11]. A subset $H_e \subseteq E(G)$ is called an edge hub set of G if every pair of edges $e, f \in E \setminus H_e$ are connected by a path where all internal edges are from H_e. The minimum cardinality of an edge hub set is called edge hub number of G, and is denoted by $h_e(G)$ [11]. An edge hub set $H_e \subseteq E(G)$ is called a connected edge hub set, if the subgraph $[H_e]$ is connected. The minimum cardinality of a connected edge hub set of G

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is called a connected edge hub number and is denoted by $h_{ce}(G)$ [1]. For more details on the hub studies we refer to [10]. Graphs G_1, and G_2 have disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Their union, $G = G_1 \cup G_2$ has, as expected, $V = V_1 \cup V2$ and $E = E_1 \cup E_2$ [6].

A set D of vertices in a graph G is called dominating set of G if every vertex in $V \setminus D$ is adjacent to some vertex in D, the minimum cardinality of a dominating set in G is called the domination number $\gamma(G)$ of a graph G ([7].

A set B of edges in a graph G is called an edge dominating set of G if every edge in $E \setminus B$ is adjacent to some edge in B, the minimum cardinality of an edge dominating set in G is called the edge domination number $\gamma'(G)$ of a graph G ([7]). An edge-domatic partition of G is a partition of $E(G)$, all of whose classes are edge-dominating sets in G. The maximum number of classes of an edge-domatic partition of G is called the edge-domatic number of G and denoted by $ed(G)$ ([1]).

A double star $S_{n,m}$ is the tree obtained from two disjoint stars $K_{1,n-1}$ and $K_{1,m-1}$ by connecting their centers [5]. The line graph $L(G)$ of G has the edges of G as it is vertices which are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G [6]. A friendship graph, is the graph obtained by taking m copies of the cycle graph C_3 with a vertex in common and denoted by F_m. The following results will be useful in the proof of our results.

Theorem 1.1([10]) *For any graph G, $h_e(G) \leq q - \Delta'(G)$, and the inequality is sharp for any path P_p, $p \geq 4$.*

Proposition 1.1([10]) *For any graph G, $h_e(G) \leq p - 3$.*

Theorem 1.2([10]) *For any tree T with $p \geq 3$ vertices and l leaves, $h_e(T) = h_{ce}(T) = p - (l + 1)$.*

Proposition 1.2([9]) *For any graph G, $\xi(G) \leq \delta(G) + 2$.*

§2. Main Results

Definition 2.1 *The maximum order of partition of the edge set $E(G)$ into edge hub sets is called edge hubtic number of G and denoted by $\xi_e(G)$. The maximum order of partition of the edge set $E(G)$ into connected edge hub sets is called connected edge hubtic number of G and denoted by $\xi_{ce}(G)$.*

It is obvious that $\xi_e(G) \geq \xi_{ce}(G)$, since $h_e(G) \leq h_{ce}(G)$. We first determine the edge hubtic number of some standard graphs.

Observation 2.1 (1) For any cycle C_p,

$$\xi_e(C_p) = \begin{cases} 3, \text{ if } p = 3 \ ; \\ 4, \text{ if } p = 4 \ ; \\ 2, \text{ if } p = 5, 6 \ ; \\ 1, \text{ if } p \geq 7. \end{cases}$$

(2) For any path P_p,

$$\xi_e(P_p) = \begin{cases} 3, \text{ if } p = 4 \ ; \\ 2, \text{ if } p = 3, 5 \ ; \\ 1, \text{ if } p \geq 6. \end{cases}$$

(3) For the wheel graph $W_{1,p-1}$, $p \geq 4$,

$$\xi_e(W_{1,p-1}) = \begin{cases} 6, \text{ if } p = 4 \ ; \\ 4, \text{ if } p = 5 \ ; \\ 3, \text{ if } p \geq 6. \end{cases}$$

(4) For the star $K_{1,p-1}$, $\xi_e(K_{1,p-1}) = p - 1$.

(5) For the double star $S_{n,m}$, $\xi_e(S_{n,m}) = 3$.

(6) For the complete bipartite graph $K_{n,m}$, $\xi_e(K_{n,m}) = max\{n, m\}$.

We will check that if the edge hubtic number is a suitable measure of stability?. Now we ask, does the edge hubtic number discriminate between graphs. There are many examples of graphs which propose that $\xi_e(G)$ is a suitable measure of stability which is able to discriminate between graphs. For example, consider the graphs G_1, G_2 and G_3 in Figure 1.

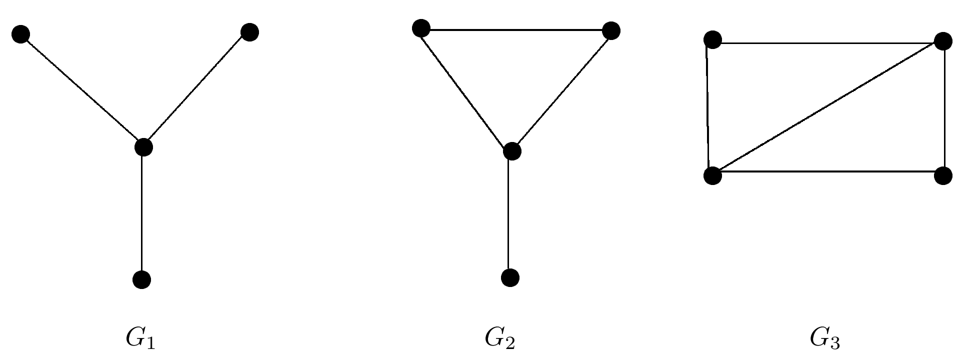

Figure 1: G_1, G_2, and G_3.

It is clear from Figure 1, that $ed(G_1) = ed(G_2) = ed(G_3) = 3$, the edge domatic number does not discriminate between graphs G_1, G_2 and G_3, but $\xi_e(G_1) = 3$, $\xi_e(G_2) = 4$ and $\xi_e(G_3) = 5$, therefore $\xi_e(G_1) \neq \xi_e(G_2) \neq \xi_e(G_3)$. So the edge hubtic number discriminates between graphs G_1, G_2 and G_3.

Observation 2.2 For any graph G, $0 \leq \xi_e(G) \leq q$.

Theorem 2.1 *If a graph G is a tree with at least 3 non-leaf edges and the induced sub graph $G[(E \setminus L)]$ is not a star where L is the set of all leaf edges in G, then $\xi_e(G) = 1$.*

Proof Let a graph G be a tree with at least 3 non-leaf edges and the induced sub graph $G[(E \setminus L)]$ is not a star, we discuss the following cases:

Case 1. Suppose that H_e is a set of all non-leaf edges, clearly any path between two leaf edges does not pass through another leaf edge. So, H_e is an edge hub set of G, and by Theorem 1.2 it is minimum edge hub set. Now, suppose $Z_e \subseteq E \setminus H_e$ be an edge hub set of G. Since G is a tree with at least 3 non-leaf edges and the induced sub graph $G[(E \setminus L)]$ is not a star, then the induced subgraph $G[E \setminus Z_e]$ is not complete. Also any path in a tree never passes through a leaf edge. Therefore there are at least two non adjacent edges $e, f \in E \setminus Z_e$ such that no path between them is in Z_e, this is a contradiction. Hence H_e is the only edge hub set.

Case 2. Suppose that H_e is an edge hub set of G but not containing all non-leaf edges. Since G has at least three non-leaf edges, let $\{e_1, e_2, e_3\}$ be non-leaf edges where e_1 and e_3 not adjacent, let l_1, l_3 be two leaf edges adjacent to e_1 and e_3, respectively. Clearly, $G[\{l_1, e_1, e_2, e_3, l_3\}]$ is a path P_6. As $h_e(P_6) = 3$, then H_e contains at least three edges from P_6. Therefore any other edge hub set of G must intersects H_e since size of P_6 is 5. Then $\xi_e(G) = 1$. □

Proposition 2.1 *For any (p, q)-graph G, $\xi_e(G) \leq \frac{q}{h_e(G)}$, where $h_e(G) \neq 0$.*

Proof Let $H = \{H_1, H_2, H_3, \cdots, H_t\}$, be the edge hubtic partition of G and $\xi_e(G) = t$. Clearly $|H_i| \geq h_e(G), i = 1, 2, \cdots, t$ and we get $q = \sum_{i=1}^{t} |H_i| \geq t h_e(G)$, hence the result. □

Observation 2.4 Let G' be a subgraph of G, then is not necessary $\xi_e(G') \leq \xi_e(G)$.

For example, $G = K_1 + P_4$, and $G' = K_1 + P_3$, $\xi_e(G') = 5 \nleq 3 = \xi_e(G)$.

Proposition 2.2 *For any (p, q)-graph G of order $p \geq 5$,*

$$\xi_e(G) \leq \delta'(G) + 2.$$

Proof By the definition of edge hub number it is obvious that $h_e(G) = h(L(G))$, so $\xi_e(G) = \xi(L(G))$. By Proposition 1.2, $\xi_e(G) = \xi(L(G)) \leq \delta(L(G)) + 2$, since $\delta'(G) = \delta(L(G))$, the result follows. □

Corollary 2.1 *For any (p, q)-graph G of order $p \geq 5$,*

$$\xi_e(G) + h_e(G) \leq \delta'(G) + p - 1.$$

Proof By Proposition 1.1 and Proposition 2.2, we get the result. □

Theorem 2.2 *For any (p,q)-graph G of order p, $\xi_e(G)+\xi_e(\overline{G}) \leq \frac{p(p-1)}{2}$, and the inequality is sharp for stars $K_{1,3}$, and $K_{1,4}$.*

Proof By Observation 2.2, $\xi_e(G) \leq q$ and $\xi_e(\overline{G}) \leq \overline{q}$. Then

$$\xi_e(G)+\xi_e(\overline{G}) \leq q+\overline{q} = \frac{p(p-1)}{2}.$$ □

Theorem 2.3 *Let G be a (p,q)-graph. Then*

$$\xi_e(G)+h_e(G) \leq q+2.$$

Proof By Theorem 1.1, $h_e(G) \leq q-\Delta'(G)$. Hence $h_e(G) \leq q-\delta'(G)$. Proposition 2.2, completes the proof. □

Observation 2.5 If $\xi_e(G_1)=\xi_e(G_2)$,then not necessary $h_e(G_1)=h_e(G_2)$.

For example, $G_1=K_{1,3}$, and $G_2=F_3$ such that $\xi_e(G_1)=\xi_e(G_2)=3$, and $h_e(G_1)=0 \neq 3=h_e(G_2)$.

Theorem 2.4 *Let G be a graph of size q. Then $\xi_e(G)=q$ if and only if G with $\delta' \geq q-2$.*

Proof Assume that $\xi_e(G)=q$, then there is a q partition of $E(G)$ into edge hub sets and every partite set consists of one edge, we have the following cases:

Case 1. All edges of G are adjacent, so any edge of G is an edge hub set of G. So $\delta'=q-1$.

Case 2. Any edge of degree $q-1$, is adjacent to all edges and hence it constitute an edge hub set of G, and since any edge of degree $q-2$, is adjacent to all edges of G except one, so every edge of them must be an edge hub set for G, hence $\delta'(G)=q-2$, if we consider any edge f such that $deg(f)<q-2$, in this case let $deg(f)=q-3$, so there is two edges e_1,e_2 not adjacent to f, now if the set $\{f\}$ is an edge hub set for G then e_1 must be adjacent to e_2, but by this assumption $\{e_1\}$ is not edge hub set for G, since e_2 not adjacent to f and e_1 not a path between them. So $\xi_e(G)=q$ only if the graph G satisfies $\delta'(G) \geq q-2$. Converse is obvious.□

Proposition 2.3 *For any two connected graphs G_1 and G_2,*

$$\xi_e(G_1 \cup G_2)=\begin{cases} 1, & \text{if } G_1 \text{ or } G_2 \text{ is with } \delta' < q-1; \\ 2, & \text{if } G_1 \text{ and } G_2 \text{ are with } \delta' = q-1. \end{cases}$$

Proof Let G_1, G_2 be two graphs both with $\delta'=q-1$, clearly $E(G_1)$ is an edge hub set for $G_1 \cup G_2$ and $E(G_2)$ is an edge hub set of the same graph, therefore $\xi_e(G_1 \cup G_2)=2$. Suppose that G_1 or G_2 is with $\delta'<q-1$, then any edge hub set of $G_1 \cup G_2$ must contain all of the edges of G_1 and any edge hub set of G_2, therefore $\xi_e(G_1 \cup G_2)=1$. □

Corollary 2.2 *For any disconnected graph G with $m \geq 3$ components, $\xi_e(G)=1$.*

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